Determining the Degree of Separation for Multiple Fractal Discharge Breakdowns in Dielectric Media Under Various Charge Distributions

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(Dated: 23 November 2017)

This paper assesses the charge separation and intermingling from a double-discharge dielectric breakdown through simulation of Poisson's equation with various charge distributions. The degree of separation between the two breakdown paths is assessed by a measure of energy potentially storable in the system. A baseline energy value for a parallel plate capacitor was calculated as E_0=7.40126. The maximum value achieved was 40.6072 in the same units, under a Gaussian charge distribution. The majority of simulations encountered a phenomena outlined called *single-breakdown domination*, that restricted results with high energies. Three of four distributions consistently gave associated energy values above the baseline for every simulation. Higher associated are expected under higher resolution

PACS numbers: 51.50.+v, 52., 52.25.Mq, 02.60.Lj
Keywords: Dielectric Breakdown, Fractals, Multiple Breakdown, Poisson's Equation

I. INTRODUCTION

The concept of dielectric breakdown is a central theme affecting phenomena from lightning to performance in solid-state electronic devices. Often, breakdown is seen as an undesirable event as in the context of transmission lines and in exceeding operating voltages in circuitry. However, the unique properties of self-avoidance, efficient spreading, and selective conduction suggest that useful devices could be fabricated based on this principle.[2]

Breakdown can be characterized as positive or negative, corresponding to electron flow out of or into the material respectively. Positive breakdown is typified by a more dendritic and self-avoiding structure and will be considered.[4] The dynamics of the breakdown can be effectively simulated by an iterative numerical analysis wherein Poisson's equation is solved at every time step to give breakdown probabilities. Two situations may be taken for boundary conditions for such a model. The first takes the breakdown path at fixed low potential and a boundary sufficiently far away at fixed high potential and solves in a charge-free medium, reducing the problem to a solution of Laplace's Equation. Here, the breakdown spreads outward freely into space like the simulation performed by Niemeyer et al.[1] The second approach considers a fixed low potential for the breakdown path, but assigns no boundary ring, instead including a spatially varying charge distribution. Using this model, a directed breakdown may be stimulated in a material. This model will be further explored.

A. Numerical Analysis of Poisson's Equation

Poisson's Equation is a generalization of Laplace's equation used to describe the electric potential field, V, in the presence of a charge distribution as well as boundary conditions.

The equation can be represented in vector-calculus notation, taking ρ to be the charge density and ε to be the electrical permittivity constant as:[5] (1)

$$\nabla^2 V = -\frac{\rho}{\varepsilon}$$

In order to perform an spatially and temporally discretized analysis, the five-point stencil method will be employed on a 2D grid. Taking arbitrary spatial units such that lattice spacing is set to 1 gives:[3] (2)

$$\nabla^2 V(x,y) = V(x+1,y) + V(x-1,y) + V(x,y+1) + V(x,y-1) - 4V(x,y)$$

The array of values for *V* can be solved by setting by concatenating columns of the 2-D array with size *(m , n)* together to give a vector with size *(m·n , 1)*. Then, a matrix *D* with size *(m·n , m·n)* can be matrix multiplied to the charge distribution at a given iteration to equate each value of the field to one quarter of the sum of each of its horizontal and vertical neighbours with edge points only considering values within the array limits. The form of *D* is shown in Figure 1. The dependence of only vertical and horizontal components, and not diagonals is justified based on the preferential breakdown in crystallographic orientations as noted by Budenstein.[2] The result of this multiplication then adds a vectorized form of the charge distribution at each point, with size *(m·n , 1)* taken in arbitrary charge units to eliminate ε. This process completes one iteration for calculating the value of *V*.

Successful numerical convergence has been found with a number of iterations greater than 30. This analysis uses 50 iterations to calculate *V*. Once the field values at each point are determined, the approach

$$\begin{bmatrix}
0 & 0.2500 & 0 & 0.2500 & 0 & 0 & 0 & 0 & 0 \\
0.2500 & 0 & 0.2500 & 0 & 0.2500 & 0 & 0 & 0 & 0 \\
0 & 0.2500 & 0 & 0 & 0 & 0.2500 & 0 & 0 & 0 \\
0.2500 & 0 & 0 & 0 & 0.2500 & 0 & 0.2500 & 0 & 0 \\
0 & 0.2500 & 0 & 0.2500 & 0 & 0.2500 & 0 & 0.2500 & 0 \\
0 & 0 & 0.2500 & 0 & 0.2500 & 0 & 0 & 0 & 0.2500 \\
0 & 0 & 0 & 0.2500 & 0 & 0 & 0 & 0.2500 & 0 \\
0 & 0 & 0 & 0 & 0.2500 & 0 & 0.2500 & 0 & 0.2500 \\
0 & 0 & 0 & 0 & 0 & 0.2500 & 0 & 0.2500 & 0
\end{bmatrix}$$

FIG. 1. The form of the matrix *D* corresponding to an array with size (3 , 3). Note the evident symmetry when the matrix is divided into 9 blocks of (3 , 3)

of Niemeyer et al. is taken, and a point is selected at random from the available points vertically or horizontally next to the breakdown paths at that time step, based on the local field values as shown:[1] (3)

$$p(i,j \to i',j') = \frac{(V_{i',j'})^{\eta}}{\sum_{i',j'}(V_{i',j'})^{\eta}}$$

The value of η is taken to be 2, and may be changed based on the given material properties.[1]

Upon selection of a given point, the point will be added to the set of breakdown-path values, from which available future points may propagate; the value of the field at that point will be set to zero; and the value of the charge distribution at that point will be set to zero. Thus the value of the field stays zero since each iteration performs: (4)

$$V_{k+1} = DV_k + C$$

Where k is the iteration number and is the C vectorized charge distribution

B. Problem Statement

The approach outlined in the previous section will be used with a starting condition where two "seeds" will be set at opposite ends of a square lattice. The values of the field and charge distribution are initially set to zero, and the points are added to the set of the breakdown-path values, so available points may originate from these two ends of the dielectric. The entirety of the rest of the dielectric will be initialized with a specified charge distribution representing excess injected electrons in the medium.

Budenstein outlines how the breakdown-path forms an electrically conductive continuous region.[4] The implication of the numerical model that has been set up is that two points are connected by a path if a breakdown has occurred from one point into the other. With this consideration, it can be seen immediately that two separate breakdown paths will never connect to form a single path since this would imply that an electrostatic force be present between two points of equal potential, and this idea is not possible by the definition of an electrostatic force being proportional to the gradient in the electric potential. In the numerical model, this prohibition is encapsulated in Equation (3), where the probability, p, becomes zero if $V_{i',j'}$ is zero. The property can be thought of as a corollary to the principle of self-avoidance for a single breakdown.

These two breakdowns will evidently be completely separated from each other, but the question may be posed as to which initial charge distributions compel the two discharges to be closest to one another. This paper seeks to simulate under different charge distributions to determine how these starting conditions produce different results.

II. CHARGE DISTRIBUTIONS

An ideal charge distribution draws each discharge toward the center for the greatest degree of closeness. Four separate charge distributions were used in the simulation of the breakdown. Under each distribution, a reflectional symmetry was preserved between the two charge distributions so that neither was favoured in breakdown probability with initial conditions. Each distribution was taken to be a simple shape and the total charge in the dielectric was normalized to 100. The units were arbitrary, but consistent between simulations.

The four distributions that were considered were:

A Flat uniform charge
B Impulse at center line
C Quadratically increasing to center maximum
D Gaussian about center

The distributions used are shown in Figure 2, where the initial seed points are located at opposite points along the variant-charge axis, halfway along the invariant-charge axis.

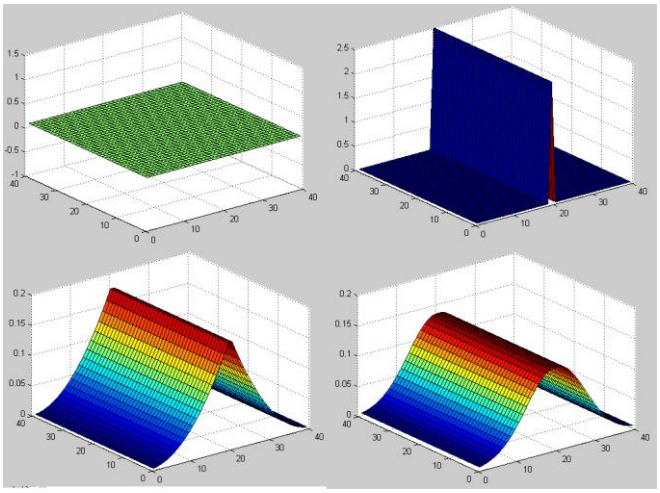

FIG. 2. The spatially varying charge distributions used in each set of simulations. Note the figures show flat distribution (top-left), impulse distribution (top-right), quadratic distribution (bottom-left) and Gaussian distribution (bottom-right)

The quadratic and Gaussian distributions are held at zero potential at each end of variant-charge axis. The Gaussian takes the form of $exp(-\alpha(x+b)^2)$, where α is set to 0.01.

III. SIMULATION SETUP

The limitations of computing power and speed provide a compromise in detail of simulation, and number of simulations for a particular charge

distribution. It was decided that adequate breakdown structural detail could be achieved, for the purpose of discerning broad characteristics of the breakdown, with a grid of size *(40 , 40)*. Computing speed rapidly drops as the number of elements in *D* increases quartically with the length of a square grid. With this size of the array, *D* has a size of *(1600 , 1600)* with *2560000* elements in it. Though it is a sparse matrix, repeated iteration becomes very slow.

For each of the given charge distributions, 200 simulations were performed in order to provide an accurate depiction of breakdown character. The simulation was started with the specified charge distributions and these distributions would be zeroed at particular breakdown-path values. For each time step in each simulation, the field value was calculated numerically using 50 iterations Equation (4), and then the next breakdown-path value was selected using Equation (3). Any particular simulation was terminated when either one quarter of the array was populated by breakdown, or when the maximum field value was below a threshold.

The outputs of each simulation was given as the indices for each breakdown path. The analysis of the data was a function of both of these path index sets as input variables

IV. RESULTS AND ANALYSIS

The results of each breakdown gave a multitude of different types of breakdowns, and therefore presented a challenge to sort through. For a majority of the data obtained, the two breakdowns were very uneven in terms of size of breakdown path. This meant that one of the seeds produced a breakdown that dominated the space of dielectric array, while the other produced a practically negligible result. Figure 3 shows examples of this feature that will be termed *single-breakdown domination*.

For an analysis of this data, a simple parameter that took the ratio of the number of elements in the larger breakdown set, divided by the number of elements in the smaller set was thus considered. This ratio, gives a value, by definition, between 0 and 1, that compares the degree of equality of the two breakdowns in size. Ratios close to 0 correspond to a single dominant path and ratios close to 1 correspond to a path that is equally spread between two breakdowns. Figure 4 gives a histogram of the ratios observed under each charge distribution.

The pervasive phenomena of single-breakdown domination can be explained by the fact that the breakdown path that starts developing closer to the charge distribution has a growing field probability at

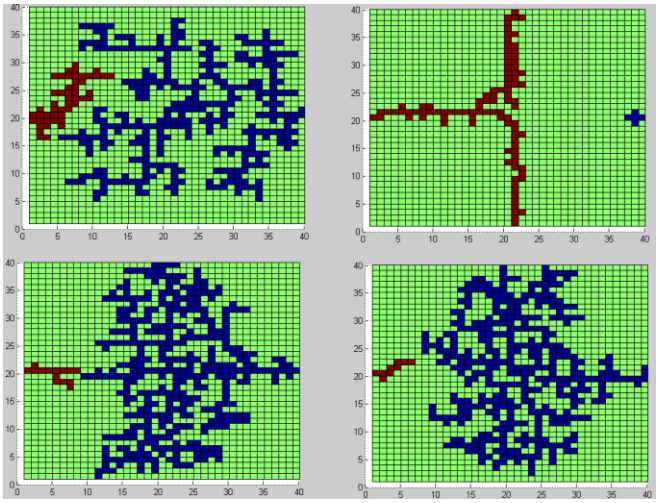

FIG. 3. Examples of Single-Breakdown Domination. This feature is inherently present in some breakdowns. The figure shows an example from each breakdown in the same order as Figure 2. The breakdowns are represented in red and blue respectively, and the rest of the dielectric is green. Shown are the cases with the smallest breakdown ratios

the tips of the breakdown (in the case of an inhomogeneous charge distribution)[4], as well as more available lattice points to spread to with each time step. The two effects together make it unlikely to have a Ratio close to 1 since the symmetry between the two breakdowns is unstable, and one breakdown quickly becomes far more likely to breakdown again than the other. The phenomena is extremely evident in charge distribution B, as shown in Figure 3, and is emblematic of the vast majority of its breakdowns.

With this analysis alone, the clear best situation would be a homogenous charge distribution that prefers an even breakdown. However the size of the breakdown isn't the only criteria. The main reason for selecting charge distributions that are heavy at the center is to draw the two breakdowns toward the

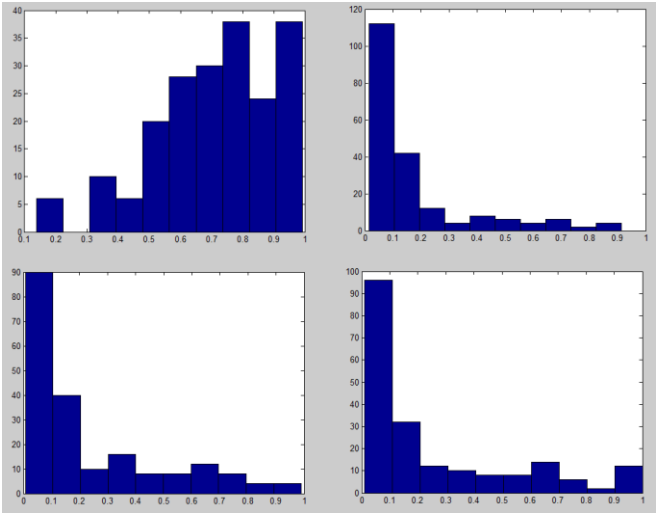

FIG. 4. Histograms of breakdown ratios in each charge distribution. The order of the charge distributions is the same as in Figure 2

center. The homogenous charge distribution gives the breakdown paths no "incentive" to move toward th center. Figure 5 shows characteristic examples of a homogenous breakdown, with the notably similar character that the two breakdowns stay on their respective sides of the dielectric, with little intermingling at the center. The fact that the distribution of the inhomogeneous case does not peak is what contributes its lack of susceptibility to single-breakdown domination, since the local field values as a breakdown moves into the dielectric actually shrinks instead of growing.

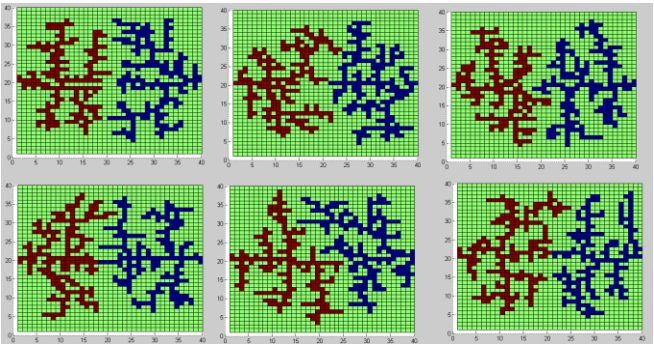

FIG. 5. High breakdown-ratio simulations for the homogenous charge distribution. Note that in this charge distribution, the two breakdowns are relatively separated

Thus the ideal breakdown structures are typically those from breakdowns B, C and D, that have breakdown ratios closer to 1. These situations can be shown to have a high degree of intermingling inherent in the fact that both breakdowns have much of their paths within the shared center region.

A complete analysis of the degree of intermingling will be quantified by a useful property. Once the breakdown is completed, the internal charge distribution will be discarded and assumed to have dissipated. Now, one breakdown will be set a positive potential and the other will be set to negative potential. The potential throughout the medium will then be calculated using Equation (4) iteratively as previously performed, however, *C* is now taken as zero. Now, the magnitude of the square of the gradient at each point is taken, and this value gives at each point the local squared electric field value by:[5] (6)

$$E = -\nabla V$$

The square of the electric field gives an associated measure of the stored energy in the field between two discharges.[5] The sum of these values over the medium is in a practical sense, a degree of intermingling between the two. The values must be tested against a baseline, so the case where two opposite edges were broken down, as in a parallel plate configuration, was tested and yielded an "energy" value of E_0 =*7.40126* in arbitrary units. Figure 6 show the analysis of the energy in a highly-intermingled breakdown that yielded energy value of *38.2498* in arbitrary (but identical) units.

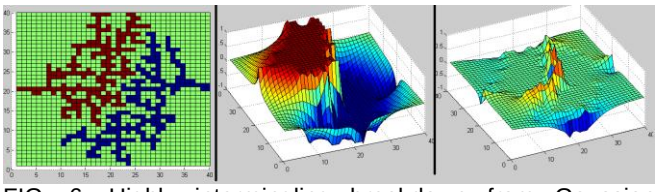

FIG. 6. Highly intermingling breakdown from Gaussian charge distribution. The left shows the breakdown paths. The middle shows the electric potential with red set to high and blue set to low. The right shows the gradient of the potential, or local electric field.

The analysis was performed on each of the performed simulations, and the cases that produced the highest stored energies were those produced from Gaussian charge distribution that had ratios close to 1. Figure 7 shows a histogram of the field energies in each simulation. The quadratic and the gaussian distributions produce the highest energies. The greatest energy calculated gives a value of *40.6077.* This is 5.496 times larger than E_0

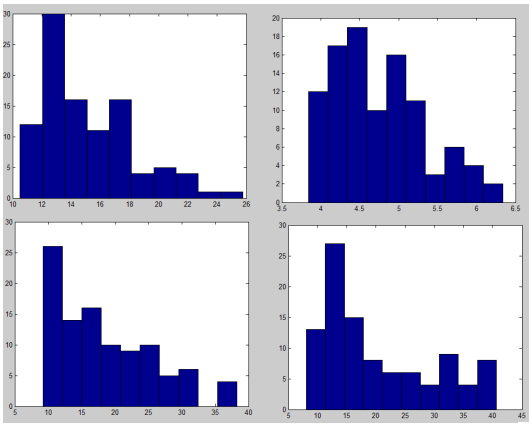

FIG. 7. Histograms of the energies for each charge distribution. The ordering is the same as in Figure 2Note the scales for each of the histograms, show that Gaussian and Quadratic give the largest results, while Impulse gives the smallest

V. Conclusion

This article simulated double breakdown in a dielectric material and used those simulations to provide useful results that could see the concept applied. Unfortunately it was evident from the data that the vast majority of the multiple breakdown events did not correspond to multiple breakdown, but instead were prone to single-breakdown domination, a concept discussed in this article. Thus it would be impractical to try to use this method to produce a physical device due to unpredictability. The best results, however managed to provide a method of stored field energy that were 5 times larger than the capacitive method at this resolution (with an energy of *40.6072*). One interesting observation is that two extremes for charge distribution were both modelled (ie. flat distribution, and impulse), and neither produced optimal results, both being well outperformed by other distributions. This means that there exists some best charge distribution for intermingling, and low separation giving high energy that has not yet been determined.

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